$$W_1(y, z) < W_2(y, z) < W_3(y, z) \qquad (5.1)$$
$$V_1(y, z) < V_2(y, z) < V_3(y, z)$$

Here the indices $n = 1, 2, 3$ characterize the dimension of system (1.1). Inequalities (5.1) are explained if we interpret the correction problem in case $n = 1$ as the problem of approaching a specified plane in a three-dimensional space at the final instant, the case $n = 2$ as a problem of approaching a straight line, and the case $n = 3$ as a problem of approaching a point. An increase in n signifies an increase in the number of correctable parameters, i.e. a complication of the control problem, and leads to a growth of the functional. Domains $D_1^\circ, \delta_1^\circ$ expand as n grows (see Fig. 3).

In conclusion we note that each solution of the correction problem, obtained in the selfsimilar variables y, z, is equivalent to the solution of an entire class of optimal impulse control problems in the original variables τ, r, q.

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OPTIMUM TRANSLATION OF A PENDULUM

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F. L. CHERNOUS'KO
(Moscow)

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A controlled mechanical system consisting of a suspended load (a pendulum), whose point of suspension can move along a horizontal straight line at some limited speed is considered. The optimum law of motion of the point of suspension is established, which ensures that the pendulum moves over a specified distance in the shortest time and is stationary at the beginning and end of translation, i.e. oscillations are absent at the end point.

This problem arises in investigations of optimum operation conditions of the widely used travelling cranes. Similar problems were previously considered, for instance, in /1 – 3/. The considered here speed of the suspension point corresponds to characteristics of a real motor, and leads to the imposition of restrictions on phase coordinates.

1. Statement of problem. The considered mechanical system consists of a physical pendulum whose suspension point M can move along the horizontal straight line Ox (Fig. 1). The following notation is used: φ is the angle of deflection of the load mass center C from vertical, x is the coordinate of the suspension point from the coordinate origin O along the x-axis, m is the mass of the load, g is the acceleration of gravity, I is the load moment of inertia about the suspension point, and L is the distance MC between the suspension point and the load center of mass. For small oscillation of a pendulum the linear equation defining its motion under the action of gravity and inertia forces is

$$I\varphi^{\cdot\cdot} = -mgL\varphi + mLw \tag{1.1}$$

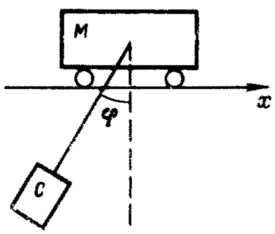

Fig. 1

where dots denote derivatives with respect to time, and w is the acceleration of the point of suspension. As stated, the velocity v of the point cannot exceed some constant value v_0. Hence

$$x^{\cdot} = v, \quad v^{\cdot} = w, \quad |v| \leqslant v_0 \tag{1.2}$$

The system begins to move at the instant of time $t = 0$ and comes to rest at some instant $t = T$. We express these conditions in the form

$$\begin{gathered}\varphi(0) = \varphi^{\cdot}(0) = x(0) = v(0) = 0 \\ \varphi(T) = \varphi^{\cdot}(T) = v(T) = 0, \quad x(T) = a\end{gathered} \tag{1.3}$$

where a is the translation of the load and the x-axis is directed so that $a \geqslant 0$. Formulas (1.1) – (1.3) represent the equations of motion of the system, the boundary conditions and the restrictions.

Equations (1.1) are valid for small oscillations and a rigid connection between the load and the point of suspension. We shall show that the considered model can be also applied to the case of a nonrigid connection similar to a thread, provided the oscillations are small.

The order of magnitude of parameters I, of the oscillation period T_0, and of the oscillation amplitude φ_0 is, respectively, mL^2, $L^{1/2}g^{-1/2}$ and $v_0 g^{-1/2}L^{-1/2}$. The condition of oscillation smallness is of the form $\varphi_0 \ll 1$. If the connection is not rigid the load can move independently of it, particularly, at the instant of change of the suspension point velocity. This results in a shock at the instant of restoration of the connection between the load and the point of suspension. It can be readily shown that the time elapsed between the instant of loss of connection between these and that of its restoration is of the order of

$$t_0 \sim v_0\varphi_0 g^{-1} \sim T_0\varphi_0^2 \ll T_0$$

Hence, if the condition $\varphi_0 \sim v_0 g^{-1/2} L^{-1/2} \ll 1$ for the smallness of oscillations is satisfied, the motion in the case of nonrigid connection is close to that of a rigid connection with a considerable degree of accuracy.

We pass to dimensionless variables, selecting v_0 as the unit of velocity and $T_0 = I^{1/2}(mgL)^{-1/2}$, which is the inverse of the natural frequency of oscillations of the pendulum. Let us make the following substitution in (1.1) – (1.3):

$$t = T_0 t', \quad x = v_0 T_0 x', \quad v = v_0 v', \quad w = v_0 T_0^{-1} w' \qquad (1.4)$$
$$\varphi = v_0 T_0^{-1} g^{-1} \varphi', \quad T = T_0 T', \quad a = v_0 T_0 a' \quad (T_0 = I^{1/2}(mgL)^{-1/2})$$

The subsequent analysis is carried out in dimensionless variables with the primes omitted for convenience. The substitution of (1.4) in formulas (1.1) and (1.2) yields

$$\varphi^{\cdot\cdot} + \varphi = w, \quad x^{\cdot} = v, \quad v^{\cdot} = w, \quad |v| \leqslant 1 \qquad (1.5)$$

while formula (1.3) remains unchanged.

1) The problem of time-optimum operation. Let distance $a > 0$ be fixed. We have to determine the law of change for $w(t)$ and for the corresponding to it $v(t)$, which would satisfy all equations (1.3) and (1.5) and yield the minimum operation time T.

2). The problem of maximum translation. Let time T of motion be fixed. We have to derive the laws for $w(t)$ and $v(t)$ which would satisfy all equations (1.3) and (1.5) and yield the maximum path a traversed by the pendulum.

Problems (1) and (2) are evidently interrelated as follows. If the solution of problem (2) yields a monotonically increasing (and it will be shown that this is so) depedence $a(T)$ of the maximum path on time, the solution of problem (1) for some $a = a_*$ is the same as that of problem (2) for $T = T_*$ derived from the equation $a(T_*) = a_*$. Because of this, problem (2) is solved first.

2. Integration of equations. The first two of Eqs. (1.5) are integrated on the assumption that $v(t)$ and $w(t)$ are specified functions and that initial conditions (1.3) are satisfied for $t = 0$. We substitute the obtained result into the boundary conditions (1.3) for $t = T$, which yields

$$\int_0^T w(\xi)\sin(\xi - T)\,d\xi = 0, \quad \int_0^T w(\xi)\cos(\xi - T)\,d\xi = 0, \qquad (2.1)$$
$$\int_0^T v(\xi)\,d\xi = a$$

To solve problem (2) it is necessary to determine functions $v(t)$ and $w(t)$, related by the equation $v^{\cdot} = w$, which satisfy Eqs. (2.1), conditions $v(T) = v(0) = 0$, and the restriction $|v(t)| \leqslant 1$ for all $0 \leqslant t \leqslant T$, and also the condition for the functional a defined by the equality in (2.1) to be maximum. All these equations, inequalities, and the functional are linear with respect to v and w. Hence the solution of the problem is valid at the limits, i.e. $|v(t)| = 1$ for almost all t's in the interval $[0, T]$.

Because of this, velocity $v(t)$ is sought in the form of a piecewise-constant function that assumes the values ± 1. We denote by n the number of nonzero intervals of constancy of $v(t)$, by t_i the length of the i-th interval, and by u the value of $v(t)$ in the first interval. Thus for the velocity and acceleration $w = v^{\cdot}$ we have

$$v(0)=v(T)=0, \quad u=\pm 1 \tag{2.2}$$

$$v(t)=u(-1)^{i+1}, \quad \sum_{j=1}^{i-1} t_j < t < \sum_{j=1}^{i} t_j, \quad i=1,\ldots,n$$

$$w(t)=u\left[\delta(t)-2\sum_{i=2}^{n}(-1)^i\delta\left(t-\sum_{j=1}^{i-1}t_j\right)+(-1)^n\delta(t-T)\right]$$

where δ denotes the delta function. Quantities t_i satisfy conditions

$$\sum_{i=1}^{n} t_i = T, \quad t_i > 0, \quad i=1,\ldots,n \tag{2.3}$$

Substituting equalities (2.2) into formulas (2.1), we obtain

$$\psi_1 \equiv \sin T - 2\sum_{i=2}^{n}(-1)^i \sin\left(\sum_{j=i}^{n} t_j\right) = 0 \tag{2.4}$$

$$\psi_2 = \cos T - 2\sum_{i=2}^{n}(-1)^i \cos\left(\sum_{j=i}^{n} t_j\right) + (-1)^n = 0$$

$$a = u\sum_{i=1}^{n}(-1)^{i+1}t_i$$

Problem (2) has, thus, been reduced to the selection of the integer n, quantity $u = \pm 1$, and of numbers t_i that satisfy restrictions (2.3) and (2.4), and also maximize quantity a defined in (2.4).

First, let us consider separately the simple cases of $n=1$ and $n=2$.

For $n=1$ the summation with respect to i is omitted in formulas (2.4) which then reduce to $\sin T=0$ and $\cos T=1$, and yield $T=2\pi k$, where k is an integer. From formula (2.3) we have $t_1=T$, and from formula (2.4) we find that $a=T$ when $u=1$. Since $|v| \leqslant 1$, we, evidently, always have $a \leqslant T$. Thus the mode with $n=1$, $t_1=T$ and $u=1$ is optimum for $T=2\pi k$ and provides to functional a the absolute maximum which is $T=2\pi k$, $k=1, 2, \ldots$ The motion of the point of suspension takes place at constant velocity $v=1$ when $0<t<T$. The phase trajectory of the pendulum in the plane φ, $\varphi^\cdot$ is shown in Fig. 2 for $T=2\pi$. The trajectory consists of segments $[0, 1]$ of the $\varphi^\cdot$-axis which is traversed in opposite directions at instants $t=0$ and $t=T$ of switching the motor on and off, and of a unit circle which for $v=1$ and $w=0$ corresponds to the pendulum motion defined by Eq. (1.5) and for $T=2\pi k$ is described k times.

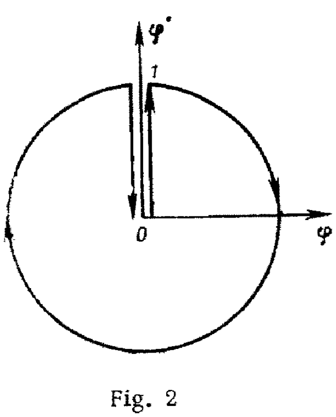

Fig. 2

For $n=2$ Eqs. (2.4) yield

$$\sin T = 2\sin t_2, \quad \cos T + 1 = 2\cos t_2.$$

Squaring and adding these two equations, after simplification, we obtain $\cos T=1$, hence $T=2\pi k$. Since the optimum control has been already determined in problem (2), hence for $T=2\pi k$ modes with $n=2$ are not optimum for any T.

In what follows we consider modes with $n>2$ and assume that

$$T=2\pi k+\tau, \quad 0<\tau<2\pi, \quad k=0, \ldots 1 \tag{2.5}$$

3. Conditions of optimum. For fixed $n>2$ the variables t_i change in the open region (see (2.3)), hence for the determination of the maximum a with restrictions (2.3) and (2.4) we formulate the Lagrange function and equate its partial derivatives to zero

$$\frac{\partial\Lambda}{\partial t_1}=u+\lambda=0, \qquad \frac{\partial\Lambda}{\partial t_s}=u(-1)^{s+1}+\lambda-2\sum_{i=2}^{s}(-1)^i\times \tag{3.1}$$

$$\left[\mu_1\cos\left(\sum_{j=i}^{n}t_j\right)-\mu_2\sin\left(\sum_{j=i}^{n}t_j\right)\right], \qquad s=2,\ldots,n$$

$$(\Lambda=a+\lambda T+\mu_1\psi_1+\mu_2\psi_2)$$

where λ, μ_1 and μ_2 are Lagrange multipliers of which the last two cannot be simultaneously zero, since then system (3.1) is not satisfied. Because of this we set $\mu_1=\mu\cos\nu$ and $\mu_2=\mu\sin\nu$, where $\mu>0$, and ν are new constants. Constructing the remainders $\partial\Lambda/\partial t_s-\partial\Lambda/\partial t_{s-1}$, after substitution, from system (3.1) we obtain

$$\cos\left(\nu+\sum_{j=s}^{n}t_j\right)=-\frac{u}{\mu}, \qquad s=2,\ldots,n \tag{3.2}$$

The subtraction of the $(s+1)$-st equation from the s-th equation of system (3.2) yields

$$\sin\frac{t_s}{2}\sin\left(\nu+\frac{t_s}{2}+\sum_{j=s+1}^{n}t_j\right)=0, \qquad s=2,\ldots,n-1 \tag{3.3}$$

Let us assume that for a certain s we have $t_s>2\pi$, and set $t_s^1=t_s-2\pi$ and $t_1^1=t_1+2\pi$ for $u=1$, and $t_s^1=t_s-2\pi$ and $t_2^1=t_2+2\pi$ for $u=-1$. In other words, we transfer intervals of 2π length from the s-th interval to the first interval for which velocity $v=1$. It will be readily seen that the above substitution does not violate conditions (2.3) and (2.4) and that quantity a which is being maximized, is not diminished by it. Thus, after the above transformation it is possible to set

$$\begin{aligned} &0<t_s<2\pi, \quad t_1>0, \quad 2\leqslant s\leqslant n, \quad u=1 \\ &0<t_s<2\pi, \quad t_2>0, \quad s\neq 2, \quad 1\leqslant s\leqslant n, \quad u=-1 \end{aligned} \tag{3.4}$$

without loss of generality.

Let us assume that for $n\geqslant 4$ and $u=-1$ we find $t_2=2\pi l_2$, where $l_2>0$ is an integer. We set $t_2^1=0$ and $t_4^1=t_4+2\pi l_2$, which is equivalent to the transfer of interval t_2 to t_4. After that the interval t_2 becomes zero, intervals t_1 and t_3 are combined, and the number n is reduced by two. These transformations do not affect parameters T and a, and can be continued until either t_2 ceases to be a multiple of 2π or $n<4$.

Because of this the equality $t_2=2\pi l_2$, $l_2=1, 2, \ldots$ need to be considered only for $n=3$. In that case the substitution of $t_2=2\pi l_2$ into Eqs. (2.4) yields

$$\sin T-2\sin t_3+2\sin t_3=0, \qquad \cos T-2\cos t_3+2\cos t_3-1=0$$

This implies that $T = 2\pi k$, where k is an integer. Since the optimum control for $T = 2\pi k$ has been already determined, it is possible to eliminate the equality $t_2 = 2\pi l_2$ from the analysis for all $n > 2$, and supplement (3.4) by

$$t_2 = 2\pi l_2 + \tau_2, \quad 0 < \tau_2 < 2\pi, \quad l_2 = 0, 1, \ldots, u = -1 \tag{3.5}$$

For $s \geqslant 2$ with conditions (3.4) and (3.5) satisfied we have $\sin(t_s / 2) \neq 0$, and Eqs. (3.3) yield

$$\nu + \frac{t_s}{2} + \sum_{j=s+1}^{n} t_j = l_s \pi, \quad s = 2, \ldots, n-1 \tag{3.6}$$

where l_s are integers. Subtracting the $(s+1)$-st from the s-th equation in (3.6), we obtain the equalities

$$t_s + t_{s+1} = 2(l_s - l_{s+1})\pi, \quad s = 2, \ldots, n-2$$

which with allowance for restrictions (3.4) and (3.5) yield for $u = 1$ and $u = -1$

$$t_s = t_2, s = 4, 6, 8, \ldots; t_s = 2\pi - t_2, s = 3, 5, 7, \ldots (s \leqslant n-1, u = -1) \tag{3.7}$$

$$t_s = \tau_2, s = 4, 6, 8, \ldots; t_s = 2\pi - \tau_2, s = 3, 5, 7, \ldots (s \leqslant n-1, u = 1) \tag{3.8}$$

Note that the transformation of system (3.1) to (3.7) and (3.8) reduces the number of equations from n to $n-3$. The omitted equations may be used for determining Lagrangian multipliers.

Each of systems (3.7) and (3.8) contain one unknown quantity (t_2 and τ_2, respectively), which makes it possible to define in its terms all t_s, except t_1 and t_n. For the determination of the four unknowns t_1, t_2 (or τ_2), t_n and a we have equalities (2.3) and (2.4) with restrictions (3.4) and (3.5). The solution that yields the maximum a is to be chosen from solutions of that system. The number n of sections and the quantity $u = \pm 1$, which are parameters affecting the solution, must also be selected so as to obtain maximum a. The modes determined in this manner are optimal.

Let us consider separately the modes with even and odd numbers of intervals $n > 2$.

4. Modes with an even number of intervals. Let us, first, set $n = 2l$, where $l > 1$ is an integer. Substituting expressions (3.7) and (3.8) into equalities (2.4) for $u = 1$ and $u = -1$, respectively, we obtain in both cases the following equations:

$$\begin{aligned} &\sin T - 2l \sin t_n + 2(l-1)\sin(t_n - t_2) = 0 \\ &\cos T - 2l \cos t_n + 2(l-1)\cos(t_n - t_2) + 1 = 0 \end{aligned} \tag{4.1}$$

where formula (3.5) is used for the case of $u = -1$. We transfer terms containing $\sin t_n$ and $\cos t_n$ to the right-hand sides of Eqs- (4.1), square both equations, and add them together. After some elementary transformations we obtain

$$\sin^2 T + (\cos T + 1)^2 + 4(l-1)^2 + 8(l-1)\cos\frac{T}{2}\cos\left(t_n - t_2 - \frac{T}{2}\right) = 4l^2$$

which after simplification becomes

$$\cos(T/2)\cos(t_n - t_2 - T/2) = 1 + (1 - \cos T)/(4l - 4) \tag{4.2}$$

The left-hand part of equality (4.2) does not exceed unity, while the right-hand part for $l > 1$ is not smaller than unity. Hence that equality is only satisfied when both of its parts are equal unity. Equating the right-hand part to unity, we obtain $\cos T = 1$

and $T = 2\pi k$, where k is an integer. Since the optimum control for $T = 2\pi k$ has been already determined, and because modes with $n > 2$ need only be considered with condition (2.5), we conclude that modes with an even number n are not optimum for any T.

5. Modes with an odd number of intervals. Let us now set $n = 2l + 1$, where $l \geqslant 1$ is an integer. Substituting conditions (3.7) and (3.8) into equalities (2.4), we obtain for both cases of $u = \pm 1$ the following equations:

$$\sin T = 2l\,[\sin(t_2 + t_n) - \sin t_n], \quad \cos T - 1 = 2l\,[\cos(t_2 + t_n) - \cos t_n] \tag{5.1}$$

The substitution into these of the expression for T in (2.5) transforms system (5.1) to

$$\sin\frac{\tau}{2}\cos\frac{\tau}{2} = 2l\sin\frac{t_2}{2}\cos\left(t_n + \frac{t_2}{2}\right), \tag{5.2}$$
$$\sin^2\frac{\tau}{2} = 2l\sin\frac{t_2}{2}\sin\left(t_n + \frac{t_2}{2}\right)$$

Since in accordance with (3.5) $\sin(t_2/2) \neq 0$, it is possible to divide these equations one by the other. We obtain

$$t_n + t_2/2 = \tau/2 + \pi r \tag{5.3}$$

where r is an integer. Substituting (5.3) into system (5.2) we obtain

$$\sin\frac{t_2}{2} = \frac{(-1)^r}{2l}\sin\frac{\tau}{2} \tag{5.4}$$

Let us introduce the notation

$$\alpha_l = \arcsin\left(\frac{1}{2l}\sin\frac{\tau}{2}\right), \quad l \geqslant 1, \quad 0 < \tau < 2\pi \tag{5.5}$$

The following inequalities

$$0 < \alpha_l \leqslant \pi/6, \quad \alpha_l < \tau/2, \quad \alpha_l < \pi - \tau/2 \tag{5.6}$$

implied by (5.5) are obvious.

First, let us find all solutions of Eqs. (5.3) and (5.4) for $u = 1$ In accordance with (3.4) we then have $0 < t_2 < 2\pi$, hence the number r in equality (5.4) must be even. The possible values of t_2 in the interval $(0, 2\pi)$ are

$$t_2' = 2\alpha_l, \quad t_2'' = 2(\pi - \alpha_l) \tag{5.7}$$

where the notation (5.5) is used. Taking into account the evenness of r and the inequalities (5.6), we obtain for t_n which correspond to (5.7) and lie in the interval $(0, 2\pi)$ from Eqs. (5.3) the following equations:

$$t_n' = \tau/2 - \alpha_l, \quad t_n'' = \pi + \tau/2 + \alpha_l \tag{5.8}$$

For $u = -1$ the solution for t_2 and t_n is derived as for $u = 1$, except that τ_2 and the integer $r - l_2$ are to be substituted for t_2 and r, respectively. Quantities τ_2 and l_2 were defined in (3.5). Similarly to (5.7) we obtain

$$\tau_2' = 2\alpha_l, \quad \tau_2'' = 2(\pi - \alpha_l) \tag{5.9}$$

As previously, t_n and t_2 are defined by formulas (5.8) and (3.5), respectively. Thus for t_2 and t_n we have two solutions (5.7) – (5.9) in each of the cases of $u = 1$ and $u = -1$, and these solutions satisfy conditions (3.4) and (3.5).

Let us now determine quantities t_1 and a. We substitute expressions (3.5),(3.7) and (3.8) for t_2 and t_s into formulas (2.3) and (2.4) for T and a. For $n = 2l + 1$ we obtain

$$t_1 + t_2 + 2(l-1)\pi + t_n = T \tag{5.10}$$
$$a = t_1 - t_2 + 2(l-1)(\pi - t_2) + t_n \qquad (u = 1)$$
$$t_1 + 2\pi l_2 + \tau_2 + 2(l-1)\pi + t_n = T$$
$$a = 2\pi l_2 + \tau_2 - t_1 - 2(l-1)(\pi - \tau_2) - t_n \qquad (u = -1)$$

Substituting expressions (2.5), (5.7) and (5.8) for T, t_2 and t_n respectively, into the first equality of (5.10), in the case of $u = 1$ we obtain for t_1 in both solutions the following equations:

$$t_1' = 2\pi(k - l + 1) + \tau/2 - \alpha_l \qquad (u = 1) \tag{5.11}$$
$$t_1'' = 2\pi(k - l + 1) + \tau/2 - 3\pi - \alpha_l$$

For $u = -1$ we similarly have for t_1 in both solutions

$$t_1' = 2\pi(k - l + 1 - l_2) + \tau/2 - \alpha_l \qquad (u = -1) \tag{5.12}$$
$$t_1'' = 2\pi(k - l + 1 - l_2) + \tau/2 - 3\pi + \alpha_l$$

For conditions $0 < t_1 < 2\pi$ to be satisfied in the case of $u = -1$ (see (3.6)) it is necessary to select the integer l_2 taking into account inequality (5.6) so that (see (5.8))

$$t_1' = t_n', \qquad t_1'' = t_n'' \qquad (u = -1) \tag{5.13}$$

Equating (5.12) and (5.13) for l_2 in both solutions, we obtain

$$l_2' = k - l + 1, \qquad l_2'' = k - l - 1 \qquad (u = -1) \tag{5.14}$$

Now in the case of $u = 1$ we substitute into the equation for a in (5.10) formulas (5.11), (5.7) and (5.8) for t_1, t_2 and t_n, respectively, and in the case of $u = -1$ formulas (5.13), (5.9), (5.14) and (5.8) for t_1, τ_2, l_2 and t_n, respectively. We then obtain for a four expressions

$$a_+' = T - 4l\alpha_l, \quad a_+'' = T - 4l(\pi - \alpha_l) \qquad (u = 1) \tag{5.15}$$
$$\left.\begin{aligned} a_-' &= T - 2\tau + 4\pi - 4l(\pi - \alpha_l) \\ a_-'' &= T - 2\tau - 4\pi - 4l\alpha_l \end{aligned}\right. \qquad (u = -1)$$

which correspond to a pair of solutions in each of the two cases of $u = \pm 1$ In (5.15) T is that defined by formula (2.5), subscripts plus and minus relate to the sign of parameter u, and the single and double primes correspond to the selected solution in formulas (5.7) – (5.9) and (5.11) – (5.14), respectively. The integer l must be chosen so that a is at its maximum.

Let us investigate the dependence of a on l. Formulas (5.5) and (5.6) imply that $\alpha_l < \pi$ and decreases with increasing l. Hence $l(\pi - \alpha_l)$ increases and a_+'' and a_-' defined in (5.15) monotonically decrease with increasing l.

The term $4l\alpha_l$ in formulas (5.15) can be represented with the use of (5.5) in the form

$$4l\alpha_l = 2(\alpha_l / \sin\alpha_l)\sin\tau/2$$

It will be readily seen that this expression monotonically increases with increasing α_l when $0 < \alpha_l < \pi/2$. Hence $4l\alpha_l$ monotonically decreases and a_+' and a_-'' defined in

(5.15) monotonically increase with increasing l.

For $u = 1$ the limits of variation of l are determined by the conditions $l \geqslant 1$ and $t_1 > 0$, while for $u = -1$ they are determined by the conditions $l \geqslant 1$ and $l_2 \geqslant 0$ From this, using equalities (5.11) and inequalities (5.6) in the case of $u = 1$ and equality (5.14) in the case of $u = -1$, we obtain the limits of variation of l in the four cases which correspond to formulas (5.15). The limits for a_+' and a_-' defined in (5.15) are $1 \leqslant l \leqslant k + 1$, and in the remaining two cases they are $1 \leqslant l \leqslant k - 1$. The last inequalities are consistent only when $k \geqslant 2$, hence solutions corresponding to a_+'' and a_-'' obtain only for $k \geqslant 2$.

Now, taking into account the monotonicity of functions (5.15) with respect to l and the limits of variation of the latter, we can determine the maximum of a in terms of l for each of the four cases. We obtain

$$\begin{aligned}
a_+' &= T - 4(k+1)\alpha_{k+1}, \quad l = k+1 \\
a_+'' &= T - 4(\pi - \alpha_1), \quad l = 1 \quad (k \geqslant 2) \\
a_-' &= T - 4(\tau/2 - \alpha_1), \quad l = 1 \\
a_-'' &= T - 4[\tau/2 + \pi + (k-1)\alpha_{k-1}], \quad l = k-1 \quad (k \geqslant 2)
\end{aligned} \tag{5.16}$$

Let us compare the functionals (5.16) for the four obtained modes. Since a_+' in (5.15) monotonically increases with increasing l, the inequality

$$a_+' \geqslant T - 4\alpha_1 = a_1 \tag{5.17}$$

is valid. Here a_1 relates to the first mode (5.15) for $l = 1$. From (5.16), (5.17) and the inequality $\alpha_1 < \pi/2$ implied by (5.6) we immediately obtain $a_+'' < a_1$.

To estimate a_-' we shall first prove that $\alpha_1 < \tau/4$. Taking the sines of both parts of equality (5.5) for $l = 1$ we obtain

$$\sin\alpha_1 = {}^1\!/_2 \sin(\tau/2) = \sin(\tau/4)\cos(\tau/4) < \sin(\tau/4)$$

Since in accordance with (5.6) $\alpha_1 < \pi/2$, it follows that $\alpha_1 < \tau/4$. The last inequality shows that parameters a_1' and a_-' defined by formulas (5.16) and (5.17) satisfy the inequality $a_-' < a_1$.

Since $k > 1$ and $\alpha_{k-1} > 0$, it follows from (5.16) that $a_-'' < a'_+$ and, consequently, $a_-'' < a_1$.

Thus the three quantities a_+'', a_-' and a_-'' defined in (5.15) are strictly smaller than $a_1 \leqslant a_+'$. Hence the first mode corresponding to $u = 1$ and to the first solution in formulas (5.7), (5.8) and (5.11) is the optimum one. This mode is superior with respect to the functional to all others, even when the choice of $l = 1$ to which corresponds functional $a_1 \leqslant a_+'$ is not the best.

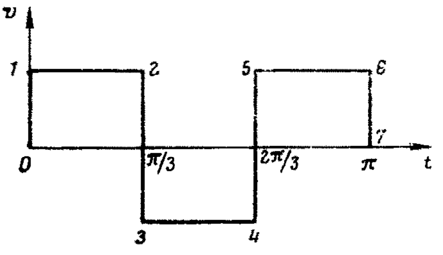

Fig. 3

6. **Analysis of optimum modes.** The solution of problem (2) is completely determined by formulas (3.9), (5.7), (5.8), (5.11) and (5.16) in which in the case of $u = 1$ quantities with a single prime apply. In accordance with the derived formulas

we have for any T of the form (2.6)

$$T = 2\pi k + \tau, \quad 0 < \tau < 2\pi, \quad l = k + 1$$
$$n = 2l + 1 = 2k + 3, \quad k = 0, 1, \ldots$$
$$t_1 = t_n = \tau / 2 - \alpha_l, \quad t_2 = t_4 = \ldots = t_{n-1} = 2\alpha_l$$
$$t_3 = t_5 = \ldots = t_{n-2} = 2(\pi - \alpha_l), \quad a = T - 4l\alpha_l \tag{6.1}$$

where α_l is that defined by (5.5). Besides the optimum solution (6.1) the first mode (5.15) is of interest in the case of $l = 1$. As indicated in Sect. 5, this mode is the best of modes with the smallest number of intervals, which is three. Setting in formulas (5.7), (5.8), (5.11) and (5.15) $l = 1$ we obtain for this mode the relationships

$$T = 2\pi k + \tau, \quad 0 < \tau < 2\pi, \quad k = 0, 1, \ldots$$
$$n = 3, \ l = 1, \ t_1 = 2\pi k + \tau / 2 - \alpha_1, \quad t_2 = 2\alpha_1$$
$$t_3 = \tau / 2 - \alpha_1, \quad a = a_1 = T - 4\alpha_1 \tag{6.2}$$

which are similar to those in (6.1). For $T < 2\pi$ and $k = 0$ mode (6.2) coincides with the optimum mode (6.1).

For $T = 2\pi k$ and $\tau = 0$ from equality (5.5) we obtain $\alpha_l = 0$. In this case solutions (6.1) and (6.2) with $n = 1$ represent the optimum solution derived above, for which

$$n = 1, \quad t_1 = T = a = 2\pi k \tag{6.3}$$

Functiona $v(t)$ and $w(t)$ for the derived modes are defined by the general formulas (2.2) into which the relationships $u = 1$ and (6.1) – (6.3) must be substituted.

As an example, let us consider the optimum motion $T = \pi$, when modes (6.1) and (6.2) are the same, and formulas (5.5) and (6.1) yield

$$T = \tau = \pi, \quad n = 3, \quad l = 1, \quad \alpha_1 = \pi/6$$
$$t_1 = t_2 = t_3 = \pi/3, \quad a = \pi/3 \tag{6.4}$$

The function $v(t)$ for the optimum motion (6.4) is shown in Fig. 3 and the pendulum

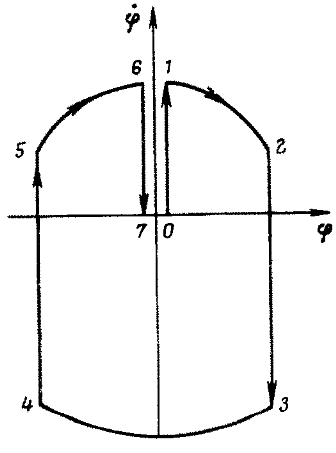

Fig. 4

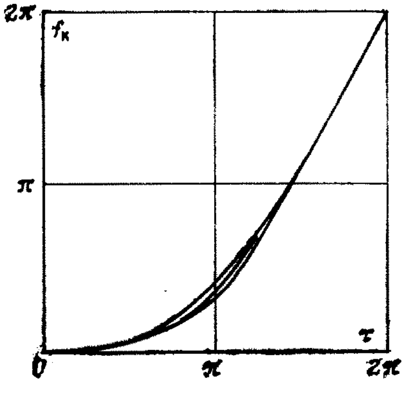

Fig. 5

phase trajectory in the $\varphi\,\varphi'$-plane is given in Fig. 4. The trajectory consists of four vertical segments which correspond to switching points. Three arcs of circles, whose centers are at the coordinate origin lie between the switching points, correspond to sections of constant velocity. The numerals zero through seven in Figs. 3 and 4 denote related points in the two diagrams. Note that the central angles of arcs in the phase plane are equal to the duration of motion along these arcs. In Fig. 5 all these angles are equal $\pi/3$.

Let us investigate functions $a(T)$ and $a_1(T)$ defined by equalities (6.1) – (6.3). Using notation (5.5) we represent these functions in the form

$$a(T) = 2\pi k + f_k(\tau), \quad T = 2\pi k + \tau, \quad 0 \leqslant \tau < 2\pi \qquad (6.5)$$
$$a_1(T) = 2\pi k + f_0(\tau), \quad k = 0, 1, 2, \ldots$$

where

$$f_k(\tau) = \tau - 4(k+1)\alpha_{k+1} = \tau - 4(k+1)\arcsin\left[\frac{\sin(\tau/2)}{2(k+1)}\right] \qquad (6.6)$$
$$0 \leqslant \tau < 2\pi, \quad k = 0, 1, \ldots$$

The curves of functions $a(T)$ and $a_1(T)$ specified by formulas (6.5) lie on the straight line $a = T$ at points $T = 2\pi k$. At remaining points we have $a_1 \leqslant a < T$. The behavior of functions $a(T)$ and $a_1(T)$ between points $T = 2\pi k$ is determined by functions $f_k(\tau)$ defined in (6.6).

A direct check will show that

$$f_k'(\tau) > 0, \quad f_k'' > 0, \quad 0 < \tau < 2\pi, \quad k = 0, 1, \ldots \qquad (6.7)$$
$$f_k(\tau) = \frac{\tau^3}{24}\left[1 - \frac{1}{4(k+1)^2}\right] + O(\tau^5), \quad \tau \to 0$$
$$f_k(\tau) = 2\pi + 2(\tau - 2\pi) + O[(\tau - 2\pi)^3], \quad \tau \to 2\pi$$

where primes denote derivatives with respect to τ.

The analysis in Sect. 5 shows that for fixed τ function $f_k(\tau)$ monotonically increases with increasing k. Passing to the limit $k \to \infty$, we obtain

$$f_0(\tau) < f_1(\tau) < \ldots < f_\infty(\tau) = \lim_{k\to\infty} f_k(\tau) = \tau - 2\sin\frac{\tau}{2}$$

Curves of function $f_k(\tau)$ are shown in Fig. 5 for $k = 0, 1$ and ∞ by the lower, middle and upper lines, respectively.

The difference between $f_0(\tau)$ and $f_\infty(\tau)$ is very small (see Fig. 5); it does not exceed 0.1, and all $f_k(\tau)$ lie between f_0 and f_∞. This implies that the simple mode (6.2) containing three constant velocity sections is very close with respect to the functional to optimum mode for all T's.

Let us consider the solution of problem (1) of time-optimum operation. Since according to (6.7) functions $f_k(\tau)$ are monotonic, functions $a(T)$ and $a_1(T)$ must also be monotonic. Therefore there exist unique inverse functions $T(a)$ and $T_1(a)$ which according to Sect. 1 provide solutions of the problem of time-optimum operation. Function $T(a)$ determines the time of the time-optimum operation and is consequently the solution of problem (1), while function $T_1(a)$ defines the time of the time-optimum operation for the class of modes with not more than three constant velocity sections. To determine $T(a)$ and $T_1(a)$ for a specified a, we set $a = 2\pi k + b$, where $0 \leqslant b < 2\pi$, $k = 0, 1, \ldots$, and then in accordance with (6.5) obtain

$$T(a) = 2\pi k + \tau, \quad \tau = f_k^{-1}(b), \qquad k = 0, 1, \ldots \qquad (6.8)$$
$$T_1(a) = 2\pi k + \tau_1, \quad \tau_1 = f_0^{-1}(b)$$

where $f_k^{-1}(\cdot)$ is a monotonic function in the interval $[0, 2\pi]$ and inverse of $f_k(\tau)$ in (6.6); it can be determined by using curves in Fig. 5. Having determined by formulas (6.8) the quantities T and T_1 for the specified a, we can calculated modes of time-optimum operation by formulas (6.1) – (6.3) and (2.3). The time-optimum operation mode (6.1), i.e. the solution of problem (1) which corresponds to T, is close with respect to the functional to the simpler mode (6.2) with three constant velocity sections, which corresponds to time T_1.

The maximum relative errors with respect to the functional, resulting from the substitution of the mode with three sections for the optimum one, does not exceed $|\Delta a / a| < 1.1\%$ in the case of problem (1) and $|\Delta T / T| < 1.2\%$ for problem (2) for any a and T.

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STUDY OF STEADY MODES OF DISTURBED AUTONOMOUS SYSTEMS IN CRITICAL CASES

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L. D. AKULENKO
(Moscow)

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A method due to Poincaré is used to study the critical cases in essentially nonlinear autonomous systems with one degree of freedom, and the situations leading to the splitting of the trajectories. The first Liapunov method is used to study the problems of stability of the steady modes. A selfrotating, almost conservative system is considered as an example. Previous papers concerned with the analysis of the motions near the generating family of periodic or rotational motions of an unperturbed system dealt, as a rule, with relatively simple cases in which the equations of the parameters of the family defining the steady mode admit, in the first approximation, simple real roots /1 – 6/. Subtler and more complex cases in which the roots are multiple, or when some of the equations of the defining system are satisfied identically, were given much less attention /1, 7 – 11/.

1. Statement of the problem. We consider a wide class of autonomous systems with one degree of freedom and slowly varying parameters of the form